The Theory of Equations

Who: Scipione del Ferro, Nicolo Fontana, Girolamo Cardano, Ludovico Ferrari, Rafael Bombelli

What: Italy's contribution to number theory

When: 1515 - 1572

By the early part of the 16th century, academic endeavors including mathematics were given increased attention. A prime example of this is the fascination with cubic and quartic equations by Italian number theorists. In 1515 Scipione del Ferro (1465-1526) algebraically solved the general cubic equation $x + mx = n$. He did not publish this result, but revealed the secret to one of his students. When, in 1535, Nicola Fontana ("Tartaglia") claimed to have discovered an algebraic solution to the cubic $x + px = n$, del Ferro's student challenged Tartaglia to a mathematical dual. In preparation for the public contest, Tartaglia diligently applied himself and only a few days before the event found an algebraic solution to cubics lacking a quadratic term. He was victorious in the contest.

In 1545, Girolamo Cardano published the great Latin treatise *Ars magna.* In it, he revealed Tartaglia's secret to solving cubics, having wrested it from Tartaglia under a solemn pledge of secrecy. A student of Cardano denied this claim, insisting that this information was received from del Ferro. The dispute became quite virulent and Tartaglia was lucky to have survived it.

An interesting byproduct of this intense push for algebraic solutions to polynomial equations was the growing exposure of Europe to imaginary numbers. Since ancient times there were always "disallowed" numbers. We have seen zero, negative numbers, and irrational numbers all progress through this stigmatized state to eventual acceptance. However, square roots of negative numbers were still taboo and "nonsense". It was Cardano who took the first peek into a world where such numbers existed. In *Ars magna*, he posed the following problem and solution.

> "Someone says to you, divide 10 into two parts, one of which multiplied into the other shall produce 40. This is impossible to solve. Nevertheless, we shall solve it in this fashion..."

He then suggested $5+\sqrt{-15}$ and $5-\sqrt{-15}$ as the two required numbers. Indeed,

$$(5+\sqrt{-15})+(5-\sqrt{-15})=10, \text{ and}$$

$$(5+\sqrt{-15})(5-\sqrt{-15})=25-5\sqrt{-15}+5\sqrt{-15}-(-15)=40.$$

Admittedly, Cardano called this solution "puzzling" and said it was "as subtle as it is useless." Fortunately, this development was far from useless.

A seemingly paradoxical situation developed soon after Cardano's controversial publication was released. A cubic equation with three real solutions was found in which Cardano's cubic formula ran across $\sqrt{-121}$. This stopped all calculations, and rendered this equation "unsolvable".

This trouble remained unresolved until Rafael Bombelli in 1572. In his text *Algebra,* he suggested that square roots of negative numbers could be introduced, at least temporarily, in solving cubic equations. In so doing, Bombelli was able to utilize $\sqrt{-1}$ and find the solution $x=4$ to the equation $x^3-15x=4$. This approach, called a "wild thought" by Bombelli himself, amazed mathematicians of the day. However, they were not convinced in the reality that such numbers existed for quite some time. In fact, Bombelli even said that this approach seemed to work by magic. In 1637, when René Descartes published *La Geometrie*, he referred to numbers such as $\sqrt{-5}$ as "imaginary". They would remain on the fringe of acceptance, yet outside of it, until the 18^{th} century when Leonhard Euler introduced the symbol i for $\sqrt{-1}$.